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On Differential Galvanometers,—
by Louis Schwendler.

(Continued from page 152, Vol. XLI, Part II, 1872.*)

The first part of this investigation concluded with the following question:

What general condition must be fulfilled in the construction of any differential galvanometer in order to make a simultaneous maximum possible with respect to an alteration of external resistance in either of the differential branches?

To answer this question, it will be necessary to remember, that the condition of a simultaneous maximum sensitiveness at or near balance was expressed by 3 equations, namely,—

$$\frac{(w-g)(w'+g')+f(w+w'+g'-g)}{p(g-w)g'}=\frac{2(g+w+f)}{2\sqrt{g}\sqrt{g'}-p(g+w)} \quad \dots \text{II}$$

$$\frac{(w'-g')(w+g)+f(w+w'+g-g')}{\frac{(g'-w')}{p}\cdot g}=\frac{2(g'+w'+f)}{2\sqrt{g}\sqrt{g'}-\frac{g'+w'}{p}} \quad \dots\dots \text{II}'$$

and

$$g'+w'-p\frac{\sqrt{g'}}{\sqrt{g}}(g+w)=0 \quad \dots\dots\dots\dots \text{I}$$

g and g' being the resistances of the two differential coils, w and w' the two resistances at which balance actually arrives, f the total resistance in the battery branch, and p an absolute number expressing what was termed the

* Read before the Asiatic Society of Bengal, 6th March, 1872.

"*mechanical arrangement*" of the differential galvanometer under consideration.

By these three equations, which are independent of each other, g, g' and p can be expressed in terms of w, w' and f.

By equation I we have at or very near balance:

$p = \frac{g'+w'}{g+w} \cdot \frac{\sqrt{g}}{\sqrt{g'}}$ which value substituted in equations II and II′ gives:

$$\frac{(w-g)(w'+g')+f(w+w'+g'-g)}{(g'+w')(g-w)g'} = \frac{2(g+w+f)}{(g'-w')(g+w)} \quad \ldots \text{II}$$

and

$$\frac{(w'-g')(w+g)+f(w+w'+g-g')}{(g+w)(g'-w')g} = \frac{2(g'+w'+f)}{(g-w)(g'+w')} \quad \ldots\ldots \text{II}'$$

and from these two equations g and g' may be developed.

This is best done by subtracting equation II from equation II′ when after reduction we get:—

$$(w'g-wg')(w'g+wg'+gg'+ww') = -f(g+g'+w+w')(w'g-wg') \quad \text{III}$$

Now it must be remembered, that with respect to our physical problem, f, w, w', g and g' represent nothing else, but electrical resistances, and that they have, therefore, to be taken in any formula as quantities of the same sign (say positive).

Consequently the above equation III would contain a mathematical impossibility (a positive quantity equal to a negative quantity), whenever the common factor $w'g-wg'$ is different from zero.

In other words equation III can only be fulfilled if we always have:

$$w'g - wg' = 0 \quad \ldots\ldots \text{IV}$$

This simple relation between the resistances at which balance arrives and the resistances of the two differential coils, expresses not only the necessary and sufficient condition under which a simultaneous maximum sensitiveness can exist, but it also affords an easy means of getting at once those special values of g, g' and p, which only solve the physical problem.

Substituting the value of either g or g', as given by equation IV in equations II and II′ and developing g and g' we have:

$$*g = -\tfrac{1}{3}\left(w + f\frac{(w+w')}{2w'}\right) + \tfrac{2}{3}\sqrt{w^2 + \frac{w}{w'}(w+w')f + \frac{(w+w')^2}{16w'^2}f^2} \ldots a.$$

$$*g' = -\tfrac{1}{3}\left(w' + f\frac{(w+w')}{2w}\right) + \tfrac{2}{3}\sqrt{w'^2 + \frac{w'}{w}(w+w')f + \frac{(w+w')^2}{16w^2}f^2} \ldots b.$$

the negative signs of the square roots having been omitted since they would

* See note at end.

obviously make g and g' negative, values which cannot solve the physical question.—

Further, if we introduce the ratio

$\frac{g'}{g} = \frac{w'}{w}$, given by equation IV, into equation I, and develope p we get:

$$p^2 = \frac{w'}{w} \text{ .... } c.$$

This latter expression shows the very simple relation which must exist between the *mechanical arrangement* of *any* differential galvanometer and the two resistances at which balance is arrived at, in order to make a simultaneous maximum sensitiveness possible.

Thus if the ratio of the two resistances at which balance arrives is fixed, the mechanical arrangement p cannot be chosen arbitrarily, but must be identical with this ratio. This is in fact the answer to the question put at the beginning of this paper.

However, the meaning of this result will be made even still clearer if we revert to equation I, by which we have

$$p \frac{\sqrt{g'}}{\sqrt{g}} = \frac{g' + w'}{g + w} = \text{C} \text{ .... I.}$$

expressing the ratio between the total resistances in the two differential branches, when balance is established, and which ratio is generally known under the name *Constant of the Differential Galvanometer.*

Substituting in the above expression I the value of $\frac{g'}{g} = \frac{w'}{w}$ from equation IV we get at once

$$\frac{w'}{w} = \text{C} \text{ .... } d.$$

and as a second answer to the question put at the beginning of this paper we have therefore:

A simultaneous maximum sensitiveness with respect to an alteration of external resistance in either branch of any differential galvanometer can be obtained only, if the constant of the differential galvanometer is equal to the ratio of the two resistances at which balance arrives, and this clearly necessitates that the resistances of the respective coils to which w and w' belong should stand in the same ratio.

The general problem may now be considered as solved by the following four general expressions:

$$g = -\tfrac{1}{3}\left(w + f\frac{(w + w')}{2w'}\right) + \tfrac{2}{3}\sqrt{w^2 + \frac{w}{w'}(w + w')f + \frac{(w + w')^2}{16w'^2}f^2} \ldots a.$$

$$g' = \frac{w'}{w} g \text{ .... } b.$$

$$p^2 = \frac{w'}{w} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad c.$$

$$C = \frac{w'}{w} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad d.$$

Additional remarks.

In the foregoing it has not been shewn that the values g and g', expressed by equations a and b, must necessarily correspond to a maximum sensitiveness of the differential galvanometer, because it was clear *à priori*, that the function by which the deflection is expressed is of such a nature that no minimum with respect to g and g' is possible. However, to complete the solution mathematically, the following is a very short proof that the values of g and g' really do correspond to a maximum sensitiveness of the differential galvanometer under consideration.

Reverting to one of the expressions for the deflection a° which any differential galvanometer gives before balance is arrived at, we had:

$a^\circ \propto K \frac{\sqrt{g}}{N} \Delta$ and as the increase of deflection at or near balance is identical with the deflection itself, and further as the law which binds the resistance of the differential coils to the other resistances in the circuit, in order to have a maximum sensitiveness, is of practical interest only when the needle is at, or very nearly at, balance, we can solve the question at once by making a° a maximum with respect to g and g', if we only suppose Δ constant and small enough, and as K is known to be independent of g and g', the deflection a° will be a maximum if $\frac{\sqrt{g}}{N}$ is a maximum for any constant Δ (zero included).

Further we know that $g' = Cg$ which value for g' in N substituted will make the latter a function of g only and consequently $\frac{\sqrt{g}}{N}$ also. We have therefore to deal with a single maximum or minimum, and according to well-known rules we have:

$$\frac{da}{dg} = \frac{N - 2g\,\frac{dN}{dg}}{2\sqrt{g}\,N^2} = \frac{U}{V}$$

and

$$\frac{d^2a}{dg^2} = \frac{V\,\frac{dU}{dg} - U\,\frac{dV}{dg}}{V^2}$$

but

$$\frac{da}{dg} = 0 \quad \text{it follows that} \quad U = 0$$

$$\therefore \qquad \frac{d^2a}{dg^2} = \frac{1}{V}\frac{dU}{dg}$$

Now

$\frac{dU}{dg} = -\left(\frac{dN}{dg} + 2g\frac{d^2N}{dg^2}\right)$, but $\frac{dN}{dg}$ as well as $\frac{d^2N}{dg^2}$ being invariably positive, it follows that $\frac{dU}{dg}$ is invariably negative, and as further V is always positive it follows finally that $\frac{d^2a}{dg^2}$ is always negative, or the value of g obtained by equation $\frac{da}{dg} = 0$ corresponds to a maximum sensitiveness of the

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equations are *not* those at which balance actually arrives, but those at which balance would arrive if no shunts were used, *i. e.*, the resistance at which balance is established when using shunts must be multiplied by the multiplying power of their respective shunts, before they are to be substituted in the equations *a*, *b*, *c* and *d*.

Mechanical arrangement designed by p.—The condition which must be fulfilled in the construction of any differential galvanometer to make a simultaneous maximum sensitiveness possible was expressed by

$$p^2 = \frac{w'}{w} \qquad \text{.......................................} \qquad c.$$

while $p = \frac{m'\,n'}{m\,\,n}$ and it will be now instructive to enquire what special physical meaning equation *c* has.

By m was understood the magnetic effect of an average convolution (*i. e.* one of average size and mean distance from the magnet acted upon, when the latter is parallel with the plane of the convolutions) in the differential coil of resistance g, when a current of unit strength passes through it. Similarly m' was the magnetic effect of an average convolution in the other differential coil of resistance g'.

Further n and n' were quantities expressed by

$$U = n\sqrt{g}$$

and

$$U' = n'\sqrt{g'}$$

U and U′ being the number of convolutions in the two coils g and g' respectively.

Now we will call A half the cross section of the coil g (cut through the coil normal to the direction of the convolutions) and which section, as the wire is to be supposed uniformly coiled, must be uniform throughout.

Thus we have generally

$$\frac{A}{c(q + \delta)} = U$$

wherever the normal cut through the coil is taken.

c is a constant indicating the manner of coiling, either by dividing the cross-section A into squares, hexagons or in any other way, but always supposing that however the coiling of the wire may have been done, it has been done uniformly throughout the coil. (This supposition is quite sufficiently nearly fulfilled in practice because the coiling should always be executed with the greatest possible care, and further the wire can be supposed practically of equal thickness throughout the coil).

q is the metallic section of the wire, and δ the non-metallic section due to the necessary insulating covering of the wire.

Further we have

$g = U\dfrac{b}{q\lambda}$ where b is the length of an average convolution and λ the absolute conductivity of the wire material supposed to be a constant for the coil.

Now, for brevity's sake, we will suppose that δ, the cross-section of the insulating covering, can be neglected against q the metallic cross-section of the wire.

Consequently we have

$$\frac{A}{cq} = U \text{ (approximately)}$$

and

$$g = U\frac{b}{q\lambda}$$

$$\therefore U = \sqrt{\frac{A\lambda}{bc}} \cdot \sqrt{g}$$

or $$n = \sqrt{\frac{A\lambda}{bc}}$$

similarly $$n' = \sqrt{\frac{A'\lambda'}{b'c'}}$$

$$\therefore \frac{n'}{n} = \sqrt{\frac{A'\lambda'\, bc}{A\lambda\, b'c'}}.$$

But using wire of the same conductivity in both the differential coils, which should be as high as is possible to procure it, and further supposing the manner of coiling to be identical in both coils, we have

$$\lambda = \lambda'$$

$$c = c'$$

$$\therefore \frac{n'}{n} = \sqrt{\frac{A'}{A} \cdot \frac{b}{b'}}$$

Further we know that if the shape and dimensions of each coil are given, and in addition also their distance from the magnet acted upon, it will be always possible to calculate m and m', though it may often present mathematical difficulties, especially if the forms of the two coils differ from each other and are also not circular. This latter condition is generally necessitated in order to obtain the greatest absolute magnetic action of each coil in as small a space as possible.

However it is clear that we may assume generally that the two coils have each an average convolution of identical shape and of the same length, placed at an equal distance from the magnet acted upon, and that therefore the magnetic action of each coil is dependent on the number of convolutions only.

In this case we have evidently

$$m = m'$$

$$b = b'$$

$$\frac{n'}{n} = \sqrt{\frac{A'}{A}}$$

and as $p = \frac{n'}{n} \cdot \frac{m'}{m}$

we have finally

$$\frac{A'}{A} = \frac{w'}{w} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad e.$$

Equation e shows at once that under the supposed conditions, *i. e.*, when the average convolutions in each coil are of equal size and shape, the wire used in either coil is of the same absolute conductivity, and that the thickness of the insulating material can be neglected against the diameter of the wire :

The wire used for filling each coil must be invariably of the same diameter, otherwise a maximum sensitiveness is impossible.

How the above simple law expressed by equation *e* would be altered, when the given suppositions were not fulfilled, must be found by further calculation, but as the latter is intricate and a more general result is not required in practice, I shall dispense at present with this labour.

Special Differential Galvanometers.—Here shall be given the special expressions to which the general equations *a, b, c* and *d*, are reduced when certain conditions are presupposed.

1*st case.*—When w and w', the two resistances at which balance is arrived at are so large that f, the resistance of the testing battery can be neglected against either of them without perceptible error. Substituting therefore $f = 0$ in equations *a*, and *b*, we get:

$$g = \frac{w}{3} \quad \ldots\ldots\ldots \quad a.$$

$$g' = \frac{w'}{3} \quad \ldots\ldots\ldots \quad b.$$

and the other two remain as they are namely:

$$p^2 = \frac{w'}{w} \quad \ldots\ldots\ldots \quad c.$$

$$C = \frac{w'}{w} \quad \ldots\ldots\ldots \quad d.$$

2*nd case.*—When the battery resistance f cannot be neglected against either w or w', but when the two resistances at which balance is arrived at are invariably equal.

Thus substituting in the general equation

$$w = w' = w$$

we get

$$g = g' = g = -\frac{w+f}{3} + \tfrac{1}{3}\sqrt{4\,w^2 + 8\,f\,w + f^2} \quad \ldots\ldots\ldots \quad a, b.$$

$$p^2 = 1 \quad \ldots\ldots\ldots \quad c.$$

$$C = 1 \quad \ldots\ldots\ldots \quad d.$$

3*rd Case.*—When the conditions given under 1 and 2 are both fulfilled

or $\quad w = w' = w$

and $\quad f = 0$

then we have

$$g = g' = g = \frac{w}{3} \quad \ldots\ldots\ldots \quad a, b.$$

$$p^2 = 1 \quad \ldots\ldots\ldots \quad c.$$

$$C = 1 \quad \ldots\ldots\ldots \quad d.$$

The very same result which was obtained by direct reasoning at the beginning of this paper.

Applications.—Though the problem in its generality has now been entirely solved, it will not perhaps be considered irrelevant to add here some applications.

For our purpose differential galvanometers may be conveniently divided into two classes, *viz.*, those in which the resistances to be measured vary within narrow limits, and those where these limits are extremely wide.

To the first class belong the differential galvanometers which are used for indicating temperature by the variation of the resistance of a metallic wire, exposed to the temperature to be measured. As for instance, C. W. Siemen's Resistance Thermometer for measuring comparatively low temperatures, or his Electric Pyrometer for measuring the high temperature in furnaces.

It is clear that for such instruments the law of maximum sensitiveness should best be fulfilled for the average resistance to be measured, which average resistance under given circumstances is always known.

To the second class belong those differential galvanometers which are used for testing Telegraph lines, at present the most important application of these instruments. In this case each differential coil should consist of separate coils connected with a commutator in such a manner that it is convenient to alter the resistance of each coil according to circumstances, *i. e.*, connecting all the separate coils in each differential coil parallel, when the resistances to be measured are comparatively low, and all the separate coils consecutively, if the resistances to be measured are high, &c., &c., fulfilling in each case the law of maximum sensitiveness for certain resistances, which are to be determined under different circumstances differently, but always bearing in mind that it is more desirable to fulfil the law of maximum sensitiveness for high resistances, when the testing current in itself is obviously weak, than for the low resistances.

An example will shew this clearer. Say for instance a differential galvanometer has to be constructed for measuring resistances between 1 and 10,000. A Siemen's comparison box of the usual kind $\left(\frac{1}{10{,}000}\right)$ being at disposal, it will be convenient and practical to decide that the two differential coils should be of equal magnetic momentum, from which it follows that C as well as p must be unity, or in other words that the two coils must be of equal size, shape and distance from the needle, and must also have equal resistances, *i. e.*, must be filled with copper wire of the same diameter. The resistance of each coil is then found by

$$g = -\frac{w+f}{3} + \tfrac{1}{3}\sqrt{4\,w^2 + 8\,f\,w + f^2}$$

where f is the resistance of the battery and w a certain value between

1 and 10,000, the two limits of measurement. The question now remains to determine w.

It is clear that the law of maximum sensitiveness has not to be fulfilled for either limit, because they represent only one of the 10,000 different resistances which have to be measured, but it is also clear that to fulfil the law for the average of the two given limits would be equally wrong, inasmuch as the maximum sensitiveness is far more required towards the highest than the lowest limit. We may assume, therefore, that it is desirable to fulfil the law for the average of the average and the highest limit, which gives

$$w = 7500$$

against which the resistance of the battery may always be neglected.

Consequently we have

$$g = \frac{w}{3} = 2500$$

for each coil.

Now if the coil be small, and consequently the wire to be used for filling it is thin, the value $g = 2500$ wants a correction to make allowance for the thickness of the insulating material, by which g becomes somewhat smaller.*

Before concluding I may remark that the question of the best resistance of the coil, when the resistance to be measured varies between two fixed or variable limits, can be solved mathematically by the application of the Variation Calculus.

* These expressions for g and g' must be corrected, if the thickness of the insulating covering of the wire cannot be neglected against its diameter. The formula by which this correction can be made was given by me in the Philosophical Magazine, January, 1866, namely

$$\text{corrected } g = c\,g\left(1 - 4\sqrt{g\,m^2}\right)$$

where g = the resistance to be corrected and expressed in Siemen's Units,

$$\text{and } m = \delta^4 \sqrt{\frac{c\,\pi\,\lambda}{AB}}$$

δ = radial thickness of the insulating covering expressed in millimetres.

c = a co-efficient expressing the arrangement adopted for filling the available space uniformly with wire. Namely, if we suppose that the cross section of the coil, by filling it up with wire, is divided into squares we have $c = 4$, if in hexagons $c = 3.4$. &c., &c.

λ = absolute conductivity of the wire material ($Hg = 1$ at freezing point).

A = half the section of the coil in question when cut normal to the direction of the convolutions, and always expressed in square millimetres.

B = length of an average convolution in the coil, and expressed in metres.